PLANE ELASTIC-PLASTIC WAVES

(O PLOSKOI UPRUGO-PLASTICHESKOI VOLNE)

PMM Vol. 29, No. 3, 1965, pp. 509-515

A.M. SKOBEEV
(Moscow)

(Received November 28, 1964)

The problem of oblique impact on a halfspace composed of material that satisfies the equations proposed by Grigorian [1] is investigated. A study is made of motions which in the elastic case correspond to longitudinal and transverse waves. If the shear modulus and the quantity $dp/d\rho$ are constant, the solution of the problem can be written down in explicit form.

Problems of soil dynamics are usually solved making the assumption that the relation between the uniaxial stress σ and the strain ϵ is given. This relation can be obtained either as an experimental result or by assuming, as in [2-5], that the pressure p is a function of the cubic dilatation θ alone and that the second invariant of the stress tensor is a function of the pressure (the plasticity condition). By considering the symmetry in the formulation of the problem, one can derive from the plasticity condition the relation $\sigma(p)$, from which $\sigma(\epsilon)$ is found. In the formulation of the present problem a stricter symmetry condition has been adopted. This involves significant complications because it leads to the fact that σ depends not only on ϵ but also on the shear stress τ. The model proposed by Grigorian [1] can be used to describe the medium. This model enables one, at least in principle, to describe an arbitrary elastic-plastic soil motion. However, the regrettable fact is that it has not been thoroughly tested. The experimental test carried out in [6] led to a concrete form of the plasticity condition and the function $p(\epsilon)$. However, the equations of this model that generalizes Hooke's law has not been tested at all, since there are no solutions of these equations available to be put to experimental test. The results of the present paper can clearly be used for testing Grigorian's model. It should be pointed out that a similar problem has been solved by Antsiferov and Rakhmatulin [7]. However, they adopted an entirely different model for soil, and the results of the present paper are significantly different from those which they obtained.

1. Let us consider an elastic-plastic halfspace and an associated system of coordinates. The y- and z-axes lie in the boundary plane and the x-axis is directed into the interior. We will use the following notation: u, and v are the velocities along the x- and y-axes, $\sigma = \sigma_{xx}$, $\tau = \sigma_{xy}$, σ_{yy} and σ_{zz} are components of the stress tensor, and

$$\gamma = {}^1/_2 (\sigma_{yy} - \sigma_{zz}), \qquad K = \rho_0 \, dp \, / \, d\rho$$

The following problem will be considered

$$\sigma = \sigma_0, \quad \tau = \tau_0, \quad \gamma = \gamma_0, \quad u = 0, \quad v = 0 \quad \text{for} \quad t < 0,\ x \geqslant 0$$
$$\sigma = \sigma_1, \quad \tau = \tau_1, \quad \gamma = \gamma_1 \quad \text{for} \quad \geqslant 0,\ x = 0$$

i.e., on the boundary $\sigma(t)$ and $\tau(t)$ are step functions. Assuming that nothing depends on y or z, the required quantities are $\sigma(x, t)$, $\tau(x, t)$, and $\gamma(x, t)$.

We will assume that the quantity $1 - \rho_0/\rho$ is small in comparison with unity. Then, the convective terms in the equation of Grigorian can be neglected, and the problem leads to the solution of the system of equations

$$\rho_0 \frac{\partial u}{\partial t} = \frac{\partial \sigma}{\partial x}, \qquad \rho_0 \frac{\partial v}{\partial t} = \frac{\partial \tau}{\partial x}, \qquad \frac{\partial \gamma}{\partial t} + \lambda\gamma = 0, \qquad \frac{\partial \tau}{\partial t} + \lambda\tau = G\frac{\partial v}{\partial x}$$
$$^3/_4 (\sigma + p)^2 + \gamma^2 + \tau^2 = F(p), \qquad \frac{\partial p}{\partial t} = -K\frac{\partial u}{\partial x} \tag{1.1}$$
$$\lambda = G\frac{\sigma + p}{F(p)}\frac{\partial u}{\partial x} + G\frac{\tau}{F(p)}\frac{\partial v}{\partial x} - \frac{F'(p)}{2F(p)}\frac{\partial p}{\partial t}$$

with the above-formulated boundary and initial conditions. The third equation in (1.1) has been obtained from the relation

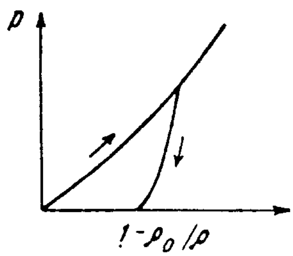

FIG. 1

$$\frac{\partial p}{\partial t} = \frac{1}{\rho_0} K \frac{\partial \rho}{\partial t} \tag{1.2}$$

and the continuity equation

$$\frac{1}{\rho_0}\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = 0 \tag{1.3}$$

The quantities $F(p)$ and λ were determined in [1].

If the shear occurs plastically, then $\lambda > 0$; in the other cases $\lambda = 0$, and system (1.1) then reduces to the usual equations in the theory of elasticity.

The first equation in system (1.1) is the equation of motion, the second equation is the generalized Hooke's law, and the third equation is the plasticity condition, where $F(p)$ is an experimentally determined plasticity function.

The pressure p is assumed to depend on ρ and the direction of the process. A typical $p(\rho)$ relation is shown in Fig. 1.

2. We will construct some solutions of system (1.1) without attempting to satisfy the initial conditions. Let us write out the equations for the characteristics of system (1.1). By definition, the characteristics of a system of first-order differential equations are

curves $x = f(t)$ having the following properties: if all the unknown functions appearing in the system are given on such a curve, the derivatives of these functions will not be uniquely defined. Denoting differention along the curve f by $\partial/\partial s$, for an arbitrary function w given on f we have

$$\frac{\partial w}{\partial s} ds = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial t} dt \tag{2.1}$$

System (1.1) contains the six unknowns u, v, σ, τ, p, and γ; the twenty partial derivatives are to be determined by the ten equations of system (1.1) and by ten equations of the type (2.1). Since the resulting system is linear in the partial derivatives, all the derivatives can be determined uniquely when the determinant is nonzero.

Thus, the condition for the characteristics $f(t)$ is the vanishing of this determinant, which has the form

$$\begin{vmatrix} dx & dt & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & dx & dt & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & dx & dt & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & dx & dt & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dx & dt & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dx & dt \\ 0 & \rho_0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ a_1^9 & 0 & a_3^9 & 0 & 0 & 0 & 0 & 0 & 0 & a_{10}^9 & 0 & 1 \\ a_1^{10} & 0 & a_3^{10} & 0 & 0 & 0 & 0 & 1 & 0 & a_{10}^{10} & 0 & 0 \\ a_1^{11} & 0 & a_3^{11} & 0 & 0 & 1 & 0 & 0 & 0 & a_{10}^{11} & 0 & 0 \\ K & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix} \tag{2.1a}$$

$$a_1^9 = G\frac{\sigma + p}{F(p)}\gamma, \qquad a_3^9 = G\frac{\tau\gamma}{F(p)}, \qquad a_{10}^9 = -\frac{F'(p)}{2F(p)}\gamma$$

$$a_1^{10} = G\frac{\sigma + p}{F(p)}\tau, \qquad a_3^{10} = G\left[\frac{\tau^2}{F(p)} - 1\right], \qquad a_{10}^{10} = -\frac{F'(p)}{2F(p)}\tau$$

$$a_1^{11} = G\left[\frac{\sigma + p^2}{F(p)} - \frac{4}{3}\right], \qquad a_3^{11} = G\frac{\tau(\sigma + p)}{F(p)}, \qquad a_{10}^{11} = -\frac{F'(p)}{2F(p)}(\sigma + p)$$

Expanding this determinant (which is not difficult because of the large number of zeros) and setting it equal to zero, we obtain the equation for the characteristics of the system

$$a^4 - \left\{\left[1 + k\left(s^2 - \frac{\gamma^2}{F}\right)^{1/2}\right]K + \frac{4 - s^2}{3}G + \frac{4\gamma^2}{3F}G\right\}a^2 + \\ + \frac{4G^2\gamma^2}{3F} + GK\left[s^2 + k\left(s^2 - \frac{\gamma^2}{F}\right)^{1/2}\right] = 0 \tag{2.2}$$

$$\rho_0\left(\frac{dx}{dt}\right)^2 = a^2, \qquad 1 - \frac{\tau^2}{F(p)} = s^2, \qquad \frac{F'(p)}{\sqrt{3F(p)}} = k$$

It should be noted [4] that $s^2 \leqslant 1$ and $k < 1$. When $\gamma \equiv 0$, equation (2.2) reduces to

$$\Phi (a^2, s) \equiv a^4 - [(1 + ks) K + {}^1/_3 (4 - s^2) G] a^2 + GKs (s + k) = 0 \quad (2.3)$$

Equation (2.2) has been obtained without any assumption whatsoever concerning the quantities K, G, and k. In the following we will assume that $G < (1 + k) K$; this assumption does not make the problem linear, because the second equation in (1.1) is nonlinear. Moreover, we will restrict ourselves to the case where $\gamma \equiv 0$ and will only consider motions for which s is constant along the characteristics. Since a^2 depends on s alone, the characteristics are straight lines.

In view of the fact that the boundary conditions have the form of step functions, the motion can be assumed to be self-modelling. Under this assumption it is possible to construct some solutions of system (1.1).

As a consequence of the self-modelling, all quantities will depend only on $x/t = a/\rho_0^{1/2}$. It is clear from (2.3) that a depends only on s, because all quantities appearing in (1.1) can be assumed to depend only on s.

From the self-modelling we have

$$\frac{\partial}{\partial t} = -\frac{a}{\sqrt{\rho_0}} \frac{\partial}{\partial x} \quad (2.4)$$

Therefore the first equation (1.1) can be written in the form

$$a^2 \frac{\partial u}{\partial x} = \frac{\partial \sigma}{\partial t} \quad (2.5)$$

Hence, making use of the third equation (1.1), we have, respectively,

$$-\frac{a^2}{K} \frac{\partial p}{\partial t} = -\frac{\partial p}{\partial t} - sk \frac{\partial p}{\partial t} - \frac{2}{\sqrt{3}} \sqrt{F} \frac{\partial s}{\partial t} \quad (2.6)$$

$$\frac{2}{\sqrt{3}} \frac{K}{a^2 - (1 + ks) K} \sqrt{F (p)} = \frac{dp}{ds} \quad (2.7)$$

or

$$\frac{d \sqrt{F}}{ds} = \frac{kK}{a^2 - (1 + ks) K} \sqrt{F} \quad (2.8)$$

Integration of (2.8) yields

$$\sqrt{F (p)} = \sqrt{F (p_0)} \exp \left(\int\limits_{s_0}^{s} \frac{kK}{a^2 (\xi) - (1 + k\xi) K} d\xi\right) \quad (2.9)$$

and according to (2.2) we have

$$\tau = \sqrt{1 - s^2} \sqrt{F (p)} \quad (2.10)$$

The remaining quantities appearing in the system can be obtained by elementary methods. Replacing a in (2.9) by the roots of equation (2.3), we obtain a family of solutions of system (1.1), which can be used for the construction of solutions of the boundary value problems.

Equation (2.3) has four roots: two are positive and two are negative. The positive ones correspond to waves propagating in the positive direction along the x-axis. Further, the solution obtained by substituting the larger positive root (2.3) into (2.9) will be called a longitudinal wave; that obtained by substituting the smaller one will be called a transverse wave.

3. Now, let us examine equations (2.3). We will assume that (2.3) is an equation in a^2. Let the larger root be denoted by a_2^2 and the smaller one by a_1^2. We will obtain an estimate of a^2 by assuming that a^2 are functions of s.

Solving (2.3), we obtain (3.1)

$$2a^2 = (1 + ks)\,K + {}^1/_3\,(4 - s^2)\,G \pm \{[(1 + ks)\,K + {}^1/_3\,(4 - s^2)\,G]^2 - 4GKs\,(s + k)\}^{1/2} \tag{3.1}$$

$$2a^2 = (1 + ks)\,K + {}^1/_3\,(4 - s^2)\,G \pm \{[(1 + ks)K - {}^1/_3\,(4 - s^2)\,G]^2 + 4GKh\}^{1/2}$$

for $s < 1$

$$h \equiv (1 + ks)\,{}^1/_3\,(4 - s^2) - s\,(s + k) = {}^1/_3\,(1 - s^2)\,(4 + ks) > 0 \tag{3.1a}$$

Therefore

$$a_1^2 < (1 + ks)\,K, \qquad a_2^2 > (1 + ks)\,K \tag{3.2}$$

This follows from the fact that for the relation

$$x_{1,2} = \alpha + \beta \pm \sqrt{(\alpha - \beta)^2 + \delta^2} \tag{3.2a}$$

we always have

$$x_1 < \min\,(\alpha,\,\beta), \qquad x_2 > \max\,(\alpha,\,\beta) \tag{3.2b}$$

Further, a very elementary analysis shows that $da_1^2/ds > 0$ and $a_1^2 \leqslant G$. One can divide the possible values of the parameters into three cases in accordance with the character of da_2^2/ds.

(1). If

$$kK > \frac{4}{3}\left(1 + \frac{4 + k}{4 + 4k}\right) G \tag{3.3}$$

then $da_2^2/ds > 0$ and $a_2^2 > K + {}^4/_3 G$ for all values of s.

(2). If

$$\frac{4}{3}\left(1 + \frac{4 + k}{4 + 4k}\right) G > kK > \frac{4}{3}\,G \tag{3.4}$$

then $a_2^2 > K + {}^4/_3 G$ and the derivative da_2^2/ds changes sign in the interval $[0,\,1]$.

(3). If

$$^4/_3 G > kK \tag{3.5}$$

then the derivative $da_2^2 / ds < 0$, $a_2^2 < K + {}^4/_3G$ when $1 > s > s_*$, where $s_* = 3kK / 4G$.

4. Now we will consider the limit of applicability of system (1.1). System (1.1) describes the motion with plastic shear. The plasticity condition for shear will be $\lambda > 0$. We will clarify when this condition can be employed. Making use of the third equation in system (1.1), we will write (2.6) in the form

$$\left[a^2 - \left(1 + \frac{k}{s}\right) K\right] \frac{\partial u}{dx} = \frac{2}{\sqrt{3}} \frac{\sqrt{1-s^2}}{s} \frac{\partial \tau}{\partial t} \qquad (4.1)$$

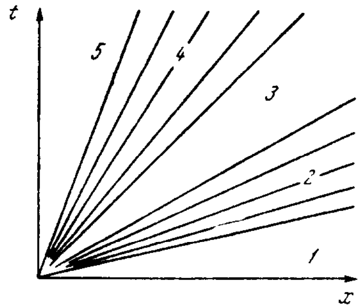

FIG. 2

With the aid of (2.4) and the first equation in (1.1), we transform the second equation in (1.1) into

$$\lambda = \frac{1}{\tau}\left(\frac{G}{a^2} - 1\right) \frac{\partial \tau}{\partial t} = \frac{\sqrt{3}}{2} \frac{(G - a^2)\,[a^2 s - (s + k)\,K]}{a^2 \tau \sqrt{1 - s^2}} \frac{\partial u}{\partial x} \qquad (4.2)$$

where account has been taken of (4.1).

By using (2.3) it is not difficult to show that (4.2) can be transformed into the form

$$\lambda = \frac{\sqrt{3}}{2s\sqrt{F}}\left[a^2 - \left(K + \frac{4}{3}G\right)\right] \frac{\partial u}{\partial x} \qquad (4.3)$$

Thus, if the material is compressed, then during loading or unloading the condition $\lambda > 0$ yields, respectively,

$$a^2 < K + {}^4/_3G \qquad (\partial u / \partial x < 0) \qquad (4.4)$$

$$a^2 > K + {}^4/_3G \qquad (\partial u / \partial x > 0) \qquad (4.5)$$

With a transverse wave $(a_1^2 \leqslant G)$ the shearing occurs plastically during loading and elastically during unloading for all values of the parameters. With longitudinal waves in cases (1), (2) and (3)($s \leqslant s_*$) the shearing occurs plastically during loading and elastically during unloading; in case (3)($s > s_*$) the shearing is elastic during unloading and plastic during loading. These results differ from the results in [4] (p. 89) where an error has been made.

5. Now we will construct the solution of the original boundary value problem by combining transverse and longitudinal waves and assuming that a uniform motion is also a solution of (1.1). We will set $\gamma \equiv 0$ and limit ourselves to the case where the loading branch in the (p, ρ)-diagram satisfied condition (3.5) and the unloading branch satisfies (3.3). In the (x, t)-plane we have the following picture: regions 1, 3, and 5 are regions of uniform movement, region 2 is the region of propagation of the longitudinal wave, and region 4 that of the transverse wave. In region 1 we have $\sigma = \sigma_0$ and $\mathcal{T} = \mathcal{T}_0$, and in region 5 we have $\sigma = \sigma_1$, and $\mathcal{T} = \mathcal{T}_1$; in region 3 we will set $\sigma = \sigma_2$, and $\mathcal{T} = \mathcal{T}_2$. Introducing the notation $[F(p)]^{1/2} = z$, the third equation in (1.1) uniquely determines the relation between σ, $\mathcal{T}$ and z, $\mathcal{T}$.

Henceforth the state of stress will be described by the quantities z, T, where we have set $z_i = [F(p_i)]^{1/2}$.

If we consider the time variation of the state of stress with a fixed value of x, we see that at first it is constant. Then, as the longitudinal wave passes, the state changes according to formulas (2.9) and (2.10) with $a^2 = a_2^2$ from z_0, T_0 to z_2, T_2. Then, it remains constant until it is changed by the transverse wave to z_1, T_1 and remains constant.

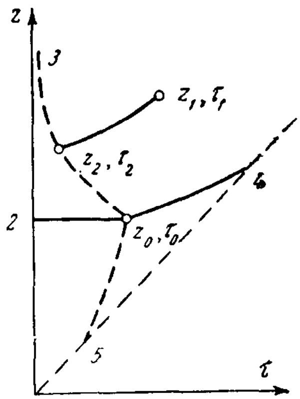

FIG. 3

Formulas (2.9) and (2.10) can be regarded as parametric representations of curves in the (z, T)-plane. There are two families of these curves. The first family corresponds to different transverse waves ($a^2 = a_1^2$) and the second family to longitudinal waves ($a^2 = a_2^2$); the choice of a curve from a family is made by fixing z_0. Let us take the point (z_0, T_0) in the (z, T)-plane and draw the curves of the first and second family passing through it. The point $z_0 T_0$ in Fig. 3 has been denoted by *1*, and the curve 1-4 represents the curve of possible passages with fixed x on a transverse compression wave, and curve 1-3 on a longitudinal compression wave, etc. The five possible cases correspond to point (z_1, T_1) lying in: (*a*) region

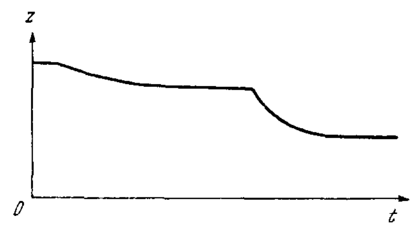

FIG. 4

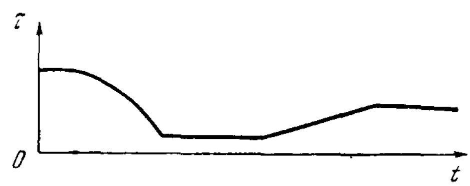

FIG. 5

(1.3.4), (*b*) in region (1.2.3), etc. When the solution of the problem exists, point (z_2, T_2) must also lie on curve 1-3 and, in addition, the curve of the second family issuing from (z_2, T_2) must pass through (z_1, T_1).

This means that

$$z_2 = z_0 \exp \Phi_2(s_0, s_2), \qquad z_1 = z_2 \exp \Phi_1(s_2, s_1)$$

$$\Phi_i(\xi, \eta) = \int_\xi^\eta \frac{kK}{a_i^2 - (1 + ks)K} ds \quad (s_2 > s_0, \ s_1 < s_2)$$

Hence we obtain the equation for s_2

$$z_0 \exp\,[\Phi_1(s_2, s_1) + \Phi_2(s_0, s_2)] = z_1 \tag{5.1}$$

When $s_2 = s_0$ the left-hand side of (5.1) is smaller than the right-hand side, provided the left-hand side determines some point on the curve 1-4 and (z_1, τ_1) lies above this curve. When $s_2 \to 1$, the left-hand side of (5.1) tends to infinity. Therefore there is an s_2 such that the left-hand side is equal to the right-hand side. This settles the matter.

Figures 4 and 5 show the dependence at a fixed station x of z and τ on time for the case considered. The remaining cases can be treated analogously. It is possible to treat other boundary value problems for system (1.1). Their solution is constructed in a similar way.

Apart from the continuous solutions of (1.1) considered in section 3, it is possible to treat discontinuous solutions corresponding to shock waves. In accordance with the stability criterion, the velocity of a shock wave must be greater than the velocities of small disturbances ahead of the wave front and less than those behind the wave front. Hence it follows that a compressive shock wave exists if $da/dp > 0$ and a rarefaction shock wave exists if $da/dp < 0$.

We have

$$\frac{da}{dp} = \frac{1}{2a}\frac{da^2}{ds}\frac{ds}{dp} \tag{5.2}$$

From (2.7) and (3.2) we have $ds/dp > 0$ an a longitudinal wave and $ds/dp < 0$ on a transverse wave. Now, from (5.2) it is clear that a shock wave can exist only in case (2) and that it will be a rarefaction wave.

In the physically important case where the loading part of the (p, ρ)-diagram satisfies condition (3.5), and the unloading part satisfies (3.3), a shock wave cannot exist.

In conclusion the author wishes to thank N.V. Zvolinskii for valuable advice and suggestions.

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Translated by D.B. McV.